

APRIL/MAY 2025

GMA12 — REAL ANALYSIS — I

Time : Three hours

Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

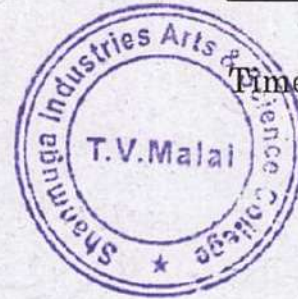
1. Define partition of $[a, b]$.
2. State the additive property of total variation.
3. State the formula for integration by parts.
4. Define step function.
5. What are the sufficient conditions for existence of Riemann – Stieltjes integrals.
6. Which is known as Bonnet's theorem?
7. Which series defines Riemann Zeta function?
8. Define convergence of double sequence $f(p, q)$.
9. State Cauchy condition for uniform convergence of series.
10. Define boundedly convergent.

19. Let f be a positive decreasing function defined on $[1, +\infty]$ such that $\lim_{x \rightarrow +\infty} f(x) = 0$. for $n=1, 2, \dots$

define $S_n = \sum_{k=1}^n f(k)$, $t_n = \int_1^n f(x) dx$, $d_n = S_n - t_n$.

Prove that the following :

- (a) $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$, for $n=1, 2, \dots$
 - (b) $\lim_{n \rightarrow \infty} d_n$ exists
 - (c) $\sum_{n=1}^{\infty} f(n)$ converges if and only if the sequence $\{t_n\}$ converges.
 - (d) $0 \leq d_k - \lim_{n \rightarrow \infty} d_n \leq f(k)$, for $k=1, 2, \dots$
20. Let α be of bounded variation on $[a, b]$. Assume that each term of the sequence $\{f_n\}$ is a real-valued function such that $f_n \in R(\alpha)$ on $[a, b]$ for each $n = 1, 2, \dots$. Assume that f_n uniformly on $[a, b]$ and define $f(x) = \int_a^x f(t) d\alpha(t)$. Prove that the following
- (a) $f \in R(\alpha)$ on $[a, b]$
 - (b) $g_n \rightarrow g$ uniformly on $[a, b]$, where $g(x) = \int_a^x f(t) d\alpha(t)$.



SECTION B — ($5 \times 5 = 25$ marks)

Answer ALL questions.

11. (a) If f is monotonic on $[a, b]$ prove that the set of discontinuities of f is countable.

Or

- (b) Let f be defined on $[a, b]$. Prove that f is of bounded variation on $[a, b]$ if and only if, f can be expressed as the difference of two increasing functions.

12. (a) State and prove Euler's summation formula.

Or

- (b) Assume that $\alpha \nearrow$ on $[a, b]$. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ and if $f(x) \leq g(x)$ for all x in $[a, b]$, then Prove that

$$\int_a^b f(x) d\alpha(x) \leq \int_a^b g(x) d\alpha(x).$$

13. (a) State and prove the second mean value theorem for Riemann stieltjes integral.

Or

- (b) Establish the theorem on change of variable in a Riemann integral.

14. (a) State and prove the Radio test.

Or

- (b) Assume that each $a_n \geq 0$. Then the product $\pi(1-a_n)$ converges if and only if, the series Σa_n converges. Prove.

15. (a) Assume that $f_n \rightarrow f$ uniformly on S . If each f_n is continuous at a point c of S , Prove that limit function f is also continuous at C .

Or

- (b) State and prove Dirichlet's test for uniform convergence.

SECTION C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

16. Let f be of bounded variation on $[a, b]$ and assume that $C \in (a, b)$. Then prove that f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have $V_f(a, b) = V_f(a, c) + V_f(c, b)$.

17. Assume that $\alpha \nearrow$ on $[a, b]$. Prove that $f \in R(\alpha)$ on $[a, b]$ if and only if given $\epsilon > 0$, we can find a partition P on $[a, b]$ such that $U[p, f, \alpha] - L[p, f, \alpha] < \epsilon$.

18. State and prove the second fundamental theorem of integral calculus.

